Physics 347 **Midterm Solutions** **Oct. 29, 2010**

Your Name: ______________________________ Your ID#:__________________________________

The Midterm will run for 50 min.

There are five problems with a total of 50 points. Each problem counts for 10 points; however, they are not of equal difficulty.

Do your work on the pages of the exam.

Please make clear how you arrive at your answers. I cannot give credit for a correct answer unless I can see how you got it. I will give partial credit for answers which are partly correct.

No books or notes of any kind are allowed. A calculator is allowed with nothing stored in memory.

A formula sheet is attached at the end of your exam. Tear it off, if you find that convenient.

I have attached an extra blank sheet (page 6) to the end of the exam in case you need extra space. If you need pages pages, let me know.

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Page 5 (10)	

TOTAL (50)

Problem 1: (10 points)

A single E. coli bacterium starts growing at noon today with unrestricted food sources. At noon two days later:

(a) Roughly how many E. coli bacteria will there be?

(b) What will be their total mass?

(c) How many C atoms will they contain?

(d) Assuming that they are growing on a glucose ($C_6H_{12}O_6$) medium, roughly what mass of glucose will have been consumed? (include metabolic energy requirements)

(a) Take the doubling time to be 50 minutes (3000 s), so the number of generations is $\frac{2\cdot 24\cdot 60}{50} = 57.6$ and the number of cells at the end of 48 hours is $2^{57.6}\boxed{= 2.18\times 10^{17}\text{ cells.}}$

(b) The mass of an E. coli is 1 pg=10^{-12} g = 10^{-15} kg, so the total mass is $\boxed{\text{218 kg.}}$

(c) Each cell has about 10^{10} C atoms, so the number of C atoms is $\boxed{2.18\times 10^{27}\text{ carbon atoms.}}$

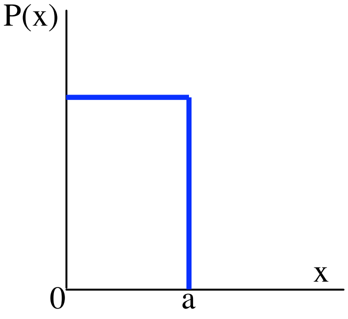
(d) The number of glucose consumed must *at least* give this number of carbons, so $\frac{2.18\times 10^{27}}{6} = 3.64\times 10^{26}$ glucose molecules. But, each glucose has MW=$6\times 12+12\times 1+6\cdot 16=180$ Da, so the total glucose mass is $3.64\times 10^{26}(180)1.67\times 10^{-27}=109$ kg. But, there is also the glucose required for metabolic energy, so, I'd estimate about double this or $\boxed{\text{218 kg}}$ glucose.

Comment:

Where did the rest of the mass come from?

Partly, from atmospheric oxygen required to metabolize the glucose for energy: Note that each C needs two O's to metabolize to CO_2 and only one of these is available from the glucose. Also, perhaps, from water, although some of the required water is generated metabolically.

Problem 2: (10 points)
A continuous random variable x is distributed uniformly over the interval [0,a] as shown at the right.



(a) What is the average <x> value of x?

(b) What is the variance σ of x?

Suppose N independent variables $\{x_n\}_{n=1}^{N}$,

each with the distribution $P(x_n)$, are added together, to form the sum $X \equiv \sum_{n=1}^{N} x_n$.

(c) What is the average value $\langle X \rangle$?

(d) What is the variance σ_N of X?

(a) You can see from the figure that the distribution is symmetrical and centered at $\boxed{\frac{a}{2}}$.
Alternatively, notice that normalization requires that the height of the distribution is 1/a.
Then, calculate: $\langle x \rangle = \frac{1}{a}\int_0^a dx\, x = \frac{1}{a}\cdot\frac{a^2}{2} = \frac{a}{2}$.

(b) $\langle x^2 \rangle = \frac{1}{a}\int_0^a dx\, x^2 = \frac{a^2}{3}$, so $\sigma^2 \equiv \langle x^2 \rangle - \langle x \rangle^2 = \frac{a^2}{3} - \left(\frac{a}{2}\right)^2 = \frac{a^2}{12} \Rightarrow \boxed{\sigma = \frac{a}{2\sqrt{3}}}$.

(c) $\langle X \rangle = N\langle x \rangle = \boxed{\frac{Na}{2}}$.

(d) $\langle X^2 \rangle = \left\langle (x_1 + x_2 + \ldots)^2 \right\rangle = N\langle x^2 \rangle + N(N-1)\langle x \rangle^2$, so

$$\sigma_N^2 \equiv \langle X^2 \rangle - \langle X \rangle^2 = N\langle x^2 \rangle + N(N-1)\langle x \rangle^2 - \left(N\langle x \rangle\right)^2 = N\sigma^2 = \frac{Na^2}{12} \text{ and } \boxed{\sigma_N = \frac{a}{2}\sqrt{\frac{N}{3}}}.$$

Problem 3. (10 points)

A three-state system has one state at energy ε_0 and two at the higher energy $\varepsilon_1 \equiv \varepsilon_0 + \Delta$. At what temperature T is the probability of finding energy ε_0 equal to that of finding ε_1?
Hint: Your answer will be proportional to Δ.

The probabilities of states at fixed T (canonical ensemble) are proportional to $e^{-\frac{E_n}{k_B T}}$. Thus the probability of finding the ground state is $\frac{1}{Z}e^{-\frac{\varepsilon_0}{k_B T}}$ while the probability of finding one of the *two* states at ε_1 is $\frac{2}{Z}e^{-\frac{\varepsilon_1}{k_B T}}$. These two become equal when $e^{-\frac{\varepsilon_0}{k_B T}} = 2e^{-\frac{\varepsilon_1}{k_B T}} \Rightarrow e^{\frac{\Delta}{k_B T}} = 2$, so $\boxed{T = \frac{\Delta}{k_B \ln 2}.}$

Problem 4: (10 points)

A statistical system s (not an ideal gas!) with N subsystems has an entropy function $S(E,N) = Ak_B\sqrt{NE}$ (A>0 is a constant).

(a) Suppose this system initially has energy E_s^0, what is its initial temperature T_s^0 (as a function of N and E_s^0)?

(b) Now consider an ideal monatomic gas of N particles in volume V with an initial energy E_g^0. What is the initial temperature T_g^0 of this gas?

(c) These two isolated systems are now put into thermal contact. What is the final temperature T^f, once the combined system comes to equilibrium?
Hint: If you do this problem right, you will end up with a quadratic equation. If you are short of time, display the equation but don't bother to solve it.

(a) The temperature is related to the energy derivative of S at constant E, so $\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{Ak_B}{2}\sqrt{\frac{N}{E}}$ and

$$\boxed{T_s^0 = \frac{2}{Ak_B}\sqrt{\frac{E_s^0}{N}}}.$$

(b) For the ideal gas, you know that $E = \frac{3}{2}Nk_BT$, so $\boxed{T_g^0 = \frac{2E_g^0}{3Nk_B}}$.

Alternatively, you can use the expression for the ideal-gas entropy, $S(E,V,N) = Nk_B \ln E^{3/2} + \text{independent of } E$, and then calculate $\frac{1}{T} = \frac{\partial S}{\partial E} = \frac{3}{2}\cdot\frac{Nk_B}{E}$, which gives the same result (of course!).

(c) The key point is that the total energy $E = E_s^0 + E_g^0$ of the isolated system cannot change, but energy will redistribute between the subsystems until the temperatures are equal. At that point

$$T_s = \frac{2}{Ak_B}\sqrt{\frac{E-E_g}{N}} = \frac{2E_g}{3Nk_B} = T_g$$

Square both sides and rearrange: $E_g^2 + \frac{9N}{A^2}E_g - \frac{9N}{A^2}E = 0$.

Solve for E_g: $E_g = \frac{N}{2}\left[-\frac{9}{A^2} + \sqrt{\left(\frac{9}{A^2}\right)^2 + \frac{36E}{A^2N}}\right]$. (the negative root is spurious)

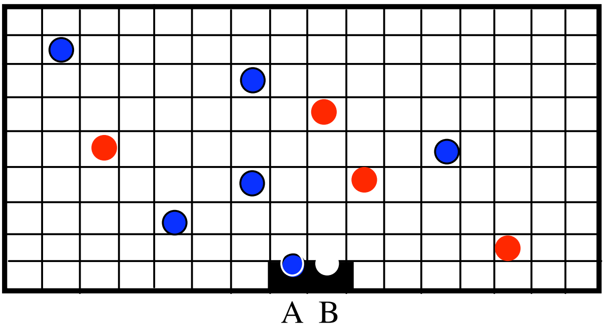
Substitute this back into the formulas for either T_g or T_s, since they have to be the same:

$$\boxed{T^f = T_s = T_g = \frac{2E_g}{3Nk_B} = \frac{3}{k_B}\left[-\frac{1}{A^2} + \sqrt{\left(\frac{1}{A^2}\right)^2 + \frac{4E}{9A^2N}}\right]}.$$

You can show that this will always lie between T_g^0 and T_s^0

Problem 5: (10 points)

Consider a receptor which R can bind ligand molecules of two types, A (blue) and B (red) but not both, i.e., if the A-site is occupied (as for, example, in the figure), then the binding of a B releases the A, and vice versa. Thus, the receptor has three states, empty (R), binding an A (RA), and binding a B (RB). I want you to consider the same kind of model of this process as used in the book and in lecture, with the bulk “solution” represented by a lattice of N_V sites. Suppose that the there are N_A molecules of type A, each of which has energy ε_S in solution and ε_A when bound and, similarly, N_B molecules of type B each of which has energy ε_S in solution and ε_B when bound. If the whole system is at temperature T, calculate the probability P_A that an A particle is bound to the receptor.



Enumerate microstates: *Three classes!*

Unbound: Energy = $(N_A + N_B)\varepsilon_s$

$$\text{Number of states} = \frac{N_V!}{N_A!N_B!(N_V - N_A - N_B)!}$$

A Bound: Energy = $(N_A + N_B - 1)\varepsilon_s + \varepsilon_A$

$$\text{Number of states} = \frac{N_V!}{(N_A-1)!N_B!(N_V - N_A - N_B + 1)!} = \frac{N_V!}{N_A!N_B!(N_V - N_A - N_B)!}\cdot\frac{N_A}{(N_V - N_A - N_B + 1)}$$

B Bound: Energy = $(N_A + N_B - 1)\varepsilon_s + \varepsilon_B$

$$\text{Number of states} = \frac{N_V!}{N_A!(N_B-1)!(N_V - N_A - N_B + 1)!} = \frac{N_V!}{N_A!N_B!(N_V - N_A - N_B)!}\cdot\frac{N_B}{(N_V - N_A - N_B + 1)}$$

The probability of A being bound is:

$$P_A = \frac{e^{-\beta((N_A+N_B-1)\varepsilon_s+\varepsilon_A)}[\#\ A\ states]}{e^{-\beta((N_A+N_B)\varepsilon_s)}[\#\ 0\ states] + e^{-\beta((N_A+N_B-1)\varepsilon_s+\varepsilon_A)}[\#\ A\ states] + e^{-\beta((N_A+N_B-1)\varepsilon_s+\varepsilon_B)}[\#\ B\ states]}$$

Divide out common factors:

$$P_A = \frac{e^{-\beta((N_A+N_B-1)\varepsilon_s+\varepsilon_A)}\dfrac{N_A}{N_V - N_A - N_B + 1}}{e^{-\beta((N_A+N_B)\varepsilon_s)} + e^{-\beta((N_A+N_B-1)\varepsilon_s+\varepsilon_A)}\dfrac{N_A}{N_V - N_A - N_B + 1} + e^{-\beta((N_A+N_B-1)\varepsilon_s+\varepsilon_B)}\dfrac{N_B}{N_V - N_A - N_B + 1}}$$

$$P_A = \frac{N_A e^{-\beta(\varepsilon_A-\varepsilon_s)}}{(N_V - N_A - N_B + 1) + N_A e^{-\beta(\varepsilon_A-\varepsilon_s)} + N_A e^{-\beta(\varepsilon_B-\varepsilon_s)}} = \boxed{\frac{c_A e^{-\beta(\varepsilon_A-\varepsilon_s)}}{(1 - c_A - c_B) + c_A e^{-\beta(\varepsilon_A-\varepsilon_s)} + c_B e^{-\beta(\varepsilon_B-\varepsilon_s)}},}$$

where $c_\alpha \equiv \dfrac{N_\alpha}{N_V}$ and the small term $1/N_v$ has been dropped.

Alternatively, we could divide by the volume $V = vN_V$ to get

$$P_A = \frac{n_A e^{-\beta(\varepsilon_A-\varepsilon_s)}}{\left(\frac{1}{v} - n_A - n_B\right) + n_A e^{-\beta(\varepsilon_A-\varepsilon_s)} + n_B e^{-\beta(\varepsilon_B-\varepsilon_s)}}$$

Extended comment:

One or two people tried to do this from the grand canonical point of view.
No one got very far with it, but I thought I’d comment.
Yes, it is entirely possible to do it this way:

Looking at the “system” states only, the grand canonical weighting of states is $P_n \sim e^{-\beta\left(E_n-\mu_A N_{A,n}-\mu_B N_{B,n}\right)}$. Note that the A and B particles need separate chemical potentials, although in this special case they happen to be the same (see below).

There are only three such states, with energies 0, ε_A, and ε_B and correspondingly $N_A = N_B = 0; N_A = 1, N_B = 0;$ and $N_A = 0, N_B = 1$.

Thus, the grand partition function (normalizing factor) is $\Xi = 1 + e^{-\beta(\varepsilon_A-\mu_A)} + e^{-\beta(\varepsilon_B-\mu_B)}$, so the probability of the A-bound state is $P_A = \dfrac{e^{-\beta(\varepsilon_A-\mu_A)}}{1 + e^{-\beta(\varepsilon_A-\mu_A)} + e^{-\beta(\varepsilon_B-\mu_B)}}$.

To go on from here, we need to find the A and B chemical potentials. This can be done is several ways, but the easiest is probably to look at the ligands in the canonical ensemble:

$F(T,N_V,N_A,N_B) = -kT\ln Z$, with $Z = \sum_n e^{-\beta E_n}$. All the microstates here have the same energy, $E_n = (N_A+N_B)\varepsilon_s$, but there are a lot of them, corresponding to the many different ways of putting N_A A’s and N_B B’s onto N_V lattice sites.

Counting them in the usual way gives a multiplicity of $\dfrac{N_V!}{N_A!N_B!(N_V-N_A-N_B)!}$.

Thus, $Z = \sum_n e^{-\beta E_n} = e^{-\beta(N_A+N_B)\varepsilon_s}\dfrac{N_V!}{N_A!N_B!(N_V-N_A-N_B)!}$.

Taking the log and using the Stirling formula gives finally,

$$F(N_A,N_B) = (N_A+N_B)\varepsilon_s + k_BT\left[N_A\ln\left(\frac{N_A}{N_V}\right) + N_B\ln\left(\frac{N_B}{N_V}\right) + (N_V-N_A-N_B)\ln\left(\frac{N_V-N_A-N_B}{N_V}\right)\right],$$

from which we can calculate $\dfrac{\partial F}{\partial N_A} = \mu_A = \varepsilon_s + k_BT\left[\ln c_A - \ln(1-c_A-c_B)\right]$, which gives $e^{\beta\mu_A} = e^{\beta\varepsilon_s}\left(\dfrac{c_A}{1-c_A-c_B}\right)$ and similarly for B. When these results are substituted back into the grand canonical formula for P_A, we get back the previous result.

Notice that in this case the chemical potential for A depends on the density of A and the density of B. This is unlike what happened for the ideal gases and comes about because in this model the A’s and B’s have an important interaction between them: they are not allowed to sit on the same lattice site.